

The exit problem from the neighborhood of a global attractor for heavy-tailed Lévy diffusions

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Abstract

We consider a finite dimensional deterministic dynamical system with a global attractor \mathcal{A} with a unique ergodic measure P concentrated on it, which is uniformly parametrized by the mean of the trajectories in a bounded set D containing \mathcal{A} . We perturb this dynamical system by a multiplicative heavy tailed Lévy noise of small intensity $\varepsilon > 0$ and solve the asymptotic first exit time and location problem from a bounded domain D around the attractor \mathcal{A} in the limit of $\varepsilon \searrow 0$. In contrast to the case of Gaussian perturbations, the exit time has the asymptotically algebraic exit rate as a function of ε , just as in the case when \mathcal{A} is a stable fixed point (see for instance [9, 18, 24]). In the small noise limit, we determine the joint law of the first time and the exit location from D^c . As an example, we study the first exit problem from a neighbourhood of a stable limit cycle for the Van der Pol oscillator perturbed by multiplicative α -stable Lévy noise.

Keywords: global attractor; regular variation; α -stable Lévy process; multiplicative noise; Itô SDE; Stratonovich SDE; canonical (Marcus) SDE; first exit time; first exit location; Van der Pol oscillator.

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1 Introduction

This article studies perturbations of finite dimensional dynamical systems by small multiplicative Lévy noise with heavy-tailed large jumps with the focus on the exit behavior from a bounded neighborhood of those global attractor. The scenario we shall study is as follows.

Let us consider a d -dimensional deterministic dynamical system $\dot{u} = f(u)$ on a positively invariant bounded domain D . We assume that the dynamical system has a global attractor \mathcal{A} in D and that uniformly over the initial conditions in D the time averages of the trajectories converge to a unique invariant measure P on \mathcal{A} . The most prominent examples of systems satisfying these settings are dynamical systems with a stable fixed point $\mathcal{A} = \{\mathfrak{s}\}$ or a stable limit cycle $\mathcal{A} = \mathcal{O}$. Clearly, in this case the paths of the dynamical system never leave D .

This situation changes significantly in the presence of a perturbation by noise, however small its intensity $\varepsilon > 0$ may be. In the generic situation, the perturbed solution always exits from D . However the growth rate of the exit time shows an asymptotic behavior that strongly depends of

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the nature of the noise. Without any doubt, beginning with the pioneering works by Kramers [21] and Freidlin and Wentzell [30], the case of Gaussian perturbations has been studied quite exhaustingly in the realm of the large deviation theory. The literature on large deviation principles is enormous and representative examples for finite and infinite dimensional systems contain the works [5, 6, 4, 7, 11, 12], where perturbed gradient dynamical systems were mainly considered. For the case of non-gradient and degenerate systems we refer to [2, 8, 13]. They all have in common that the first exit time rate grows in ε with the order $\exp(\bar{V}/\varepsilon^2)$, in physics literature known as Kramer's law, where \bar{V} is the minimal amount of energy needed for a Brownian path to steer the perturbed system from the attractor \mathcal{A} to a point on the boundary ∂D . In other words, \bar{V} depends only on the dynamical system outside the attractor. The dynamics on the attractor, where no energy is needed to travel, is irrelevant.

The exit scenario changes fundamentally if the perturbation is a Lévy process, with power tailed (heavy tailed) large jumps. In this case, the large jumps determine the exit behavior: It is possible to perform a time scales separation of big jumps from the small jumps and the Gaussian component such that on the new time scale the system's small noise behavior becomes essentially one of a deterministic system perturbed by large jumps. Using this approach, the gradient case or the case with point attractors in finite and infinite dimensional systems has been treated in [9, 14, 18, 19, 24]. Since the deterministic system converges to the stable state fast enough in comparison to the occurrence rate of large jumps, the exit occurs when a system jumps from a vicinity of \mathfrak{s} . The resulting exit rate turns out to be of a power order with respect to $1/\varepsilon$, and the asymptotic exit location in D^c is given by the probability distribution of large jump increments conditioned to $D^c - \mathfrak{s}$. This is radically different from the case of Gaussian perturbations, where the exit occurs only on the boundary of D due to the continuity of the paths.

In the present paper, we generalize these results to the case where the global attractor \mathcal{A} in D is not necessarily a stable point. Once again the essential exit behavior is determined by the deterministic system perturbed by large jumps. However we face the problem that — opposite to the case of gradient systems — the convergence of the deterministic trajectory to a hyperbolic attractor as a set does not imply the convergence towards a trajectory on the attractor. Instead, what replaces the deterministic control of the trajectory is its ergodic behavior, that is its “occupation statistics” of its time-average on the attractor. In this sense the exit event will be asymptotically triggered by the large jumps starting on \mathcal{A} under the invariant measure P . The exit rate is again of a power order in $1/\varepsilon$, but the precise prefactor depends now on the large jump distribution and the ergodic measure P concentrated on \mathcal{A} . The distribution of the exit location is hence given by the probability distribution for large jumps conditioned to $D - v$, where v is averaged over P on \mathcal{A} . Therefore contrary to the aforementioned Gaussian case, the deterministic dynamics on the attractor turns out to be crucial for the asymptotics of the exit times.

We can make this intuition rigorous for a very general class of additive and multiplicative Lévy noises with a regularly varying Lévy measure of index $-\alpha$, $\alpha > 0$. In particular, our main result covers perturbations in Itô and Stratonovich, as well as in the canonical (Marcus) integrals sense, where jumps in general do not occur along straight lines, but follow the flow of a vector field which determines the multiplicative noise.

Limit cycles attractors perturbed by Gaussian noise are considered in the physics and other natural sciences literature since quite some time [10, 15, 16, 22, 23, 27]. As an application of our main result we work out the example of the Van der Pol oscillator perturbed by multiplicative α -stable noise.

It has been well-known for a long time, that the first exit time and location problem for general Markov processes can be stated in terms a Poisson and Dirichlet problem of the generator of this process, consult for instance [31]. However, the generators of the jump part in the case of Lévy

processes are non-local integro-differential operators, for which these problems are hard to solve, in particular in the case of the canonical Marcus noise. The advantage of our approach is among others the insensitivity to the boundary regularity of D and the intuitive simplicity of the result.

2 Object of study and main result

2.1 Deterministic dynamics

Consider a bounded domain $D \subset \mathbb{R}^d, d \geq 1$ with piecewise \mathcal{C}^1 -smooth boundary and a vector field $f \in \mathcal{C}^2(D, \mathbb{R}^d)$, which points uniformly inward at the boundary. We are interested in the d -dimensional dynamical system given as the solution map $(t, x) \mapsto u(t; x)$, $t \geq 0$, $x \in D$ of the autonomous ordinary differential equation

$$\dot{u} = f(u), \quad u(0) = x. \quad (1)$$

We further assume that the unique solution exists for all $x \in D$ and $t \geq 0$. Further we assume that the dynamical system defined by (1) has a global attractor \mathcal{A} in D .

Remark 2.1 *Since by definition the global attractor attracts bounded sets in D , see for instance [29], there exists a positively invariant set \mathcal{I} with $\mathcal{A} \subset \mathcal{I} \subset D$ such that $\text{dist}(\partial D, \partial \mathcal{I}) > 0$ for which there is a time $\mathcal{S} > 0$ such that for all $x \in D$ and $t \geq \mathcal{S}$*

$$u(t; x) \in \mathcal{I}. \quad (2)$$

(D.1) Let there exist a unique invariant probability measure P on $\mathfrak{B}(\mathbb{R}^d)$ with $\text{supp}(P) = \mathcal{A}$ such that all non-negative, measurable and bounded functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy

$$\lim_{t \rightarrow \infty} \sup_{x \in D} \frac{1}{t} \int_0^t \varphi(u(s; x)) ds = \int_{\mathcal{A}} \varphi(v) P(dv). \quad (3)$$

Definition 2.1 For $\delta > 0$ we define the reduced domain of attraction

$$D_\delta := D \setminus \mathcal{B}_\delta(\partial D).$$

Remark 2.2 Due to the assumption on the uniform inward pointing of f at ∂D , there is $\delta_0 \in (0, 1)$ such that for all $\delta \in (0, \delta_0]$

$$u(t, D_\delta) \subset D_\delta \quad \text{for all } t \geq 0.$$

2.2 The probabilistic perturbation

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, satisfying the usual hypothesis in the sense of [25], we consider a Lévy process $Z = (Z_t)_{t \geq 0}$ with values in \mathbb{R}^m , $m \geq 1$ and the following characteristic function

$$\mathbf{E} e^{i\langle u, Z_1 \rangle} = \exp \left(-\frac{\langle Au, u \rangle}{2} + i\langle b, u \rangle + \int (e^{i\langle u, z \rangle} - 1 - i\langle u, z \rangle \mathbf{1}(\|z\| \geq 1)) \nu(dz) \right), \quad u \in \mathbb{R}^m. \quad (4)$$

Let us denote by $N(dt, dz)$ the associated Poisson random measure with the intensity measure $dt \otimes \nu(dz)$ and the compensated Poisson random measure $\tilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz)$. Consequently, by the Lévy–Itô theorem [1] the Lévy process Z given above has the following almost surely pathwise additive decomposition

$$Z_t = bt + A^{\frac{1}{2}} B_t + \int_{(0, t]} \int_{0 < \|z\| < 1} z \tilde{N}(ds, dz) + \int_{(0, t]} \int_{\|z\| \geq 1} z N(ds, dz), \quad t \geq 0, \quad (5)$$

with $B = (B_t)_{t \geq 0}$ a standard Brownian motion in \mathbb{R}^m . Furthermore, the random summands in (5) are independent. For further details on Lévy processes we refer to [1] and [28].

(S.1) The Lévy measure ν of the process Z is **regularly varying at ∞** with index $-\alpha$. Let $h: (0, \infty) \rightarrow (0, \infty)$ denote its tail,

$$h(r) := \int_{\|y\| \geq r} \nu(dy). \quad (6)$$

Then there exist $\alpha > 0$ and a non-trivial self-similar Radon measure μ on $\bar{\mathbb{R}}^m \setminus \{0\}$ such that $\mu(\bar{\mathbb{R}}^m \setminus \mathbb{R}^m) = 0$ and for any $a > 0$ and any Borel set A bounded away from the origin, $0 \notin \bar{A}$ with $\mu(\partial A) = 0$, the following limit holds true:

$$\mu(aA) = \lim_{r \rightarrow \infty} \frac{\nu(raA)}{h(r)} = \frac{1}{a^\alpha} \lim_{r \rightarrow \infty} \frac{\nu(rA)}{h(r)} = \frac{1}{a^\alpha} \mu(A). \quad (7)$$

In particular, following [3] there exists a positive function ℓ slowly varying at infinity such that

$$h(r) = \frac{1}{r^\alpha \ell(r)}, \quad \text{for all } r > 0.$$

The selfsimilarity property of the limit measure μ implies that μ assigns no mass to spheres centred at the origin of \mathbb{R}^m and has no atoms. For more information on multivariate heavy tails and regular variation we refer the reader to Hult and Lindskog [17] and Resnick [26].

(S.2) Consider continuous maps $G \in \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^m, \mathbb{R}^d)$ and $F, H : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and fix the notation

$$a(x, y) := F(x)F(y)^T \quad \text{for } x, y \in \mathbb{R}^d.$$

We assume that there exists $L > 0$ such that f, G, H and F satisfy the following properties.

1. **Local Lipschitz conditions:** For all $x, y \in D$

$$\begin{aligned} & \|f(x) - f(y)\|^2 + \|a(x, x) - 2a(x, y) + a(y, y)\| + \|H(x) - H(y)\|^2 \\ & + \|F(x) - F(y)\|^2 + \int_{\mathcal{B}_1} \|G(x, w) - G(y, w)\|^2 \nu(dw) \leq L^2 \|x - y\|^2. \end{aligned}$$

2. **Local boundedness:** For all $x \in D$

$$\|f(x)\|^2 + \|a(x, x)\| + \|H(x)\|^2 + \|F(x)\|^2 + \int_{\mathcal{B}_1} \|G(x, w)\|^2 \nu(dw) \leq L^2 (1 + \|x\|^2).$$

3. **Large jump coefficient:** For all $x, y \in D$ and $w \in \mathbb{R}^m$

$$\|G(x, w) - G(y, w)\| \leq L e^{L(\|w\| \wedge L)} \|x - y\|.$$

4. **Local bound for G in small balls:** There exists $\delta' > 0$ such that for $w \in \mathcal{B}_{\delta'}(0)$

$$\sup_{v \in \mathcal{B}_{\delta'}(\mathcal{A})} \|G(v, w)\| \leq L.$$

Proposition 2.1 *Let the assumptions (D.1) and (S.1-2) be fulfilled. Then for $\varepsilon \in (0, 1)$ and $x \in D$ the stochastic differential equation*

$$\begin{aligned} X_{t,x}^\varepsilon = & x + \int_0^t f(X_{s,x}^\varepsilon) ds + \varepsilon \int_0^t H(X_{s,x}^\varepsilon) b ds + \varepsilon \int_0^t F(X_{s,x}^\varepsilon) d(A^{\frac{1}{2}} B_s) \\ & + \int_0^t \int_{\|z\| \leq 1} G(X_{s-,x}^\varepsilon, \varepsilon z) \tilde{N}(ds, dz) + \int_0^t \int_{\|z\| > 1} G(X_{s-,x}^\varepsilon, \varepsilon z) N(ds, dz). \end{aligned} \quad (8)$$

has a unique local strong solution process $(X_{t \wedge \mathbb{T}, x}^\varepsilon)_{t \geq 0}$ with càdlàg paths in \mathbb{R}^d and defines a strong Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$, where $\mathbb{T} = \mathbb{T}_x(\varepsilon)$ is the first exit time

$$\mathbb{T}_x(\varepsilon) := \inf\{t \geq 0 : X_{t,x}^\varepsilon \notin D\}, \quad \varepsilon > 0, x \in D.$$

The proof can be found for instance stated as Theorem 6.23 in [1] on page 367.

The multiplicative perturbations in the sense of Itô, Fisk–Stratonovich or Marcus are of a special interest for applications. Assume that Z is a pure jump process with $A = 0$, $b = 0$. For a globally Lipschitz continuous function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ consider the Itô and canonical SDEs

$$X_t = x + \int_0^t f(X_s) dt + \varepsilon \int_0^t \Phi(X_{s-}) dZ_s, \quad (9)$$

$$X_t^\diamond = x + \int_0^t f(X_s^\diamond) dt + \varepsilon \int_0^t \Phi(X_{s-}^\diamond) \diamond dZ_s. \quad (10)$$

Then the Itô SDE (9) is obtained from (8) with

$$G(x, z) := x - \Phi(x)z$$

and the Marcus SDE (10) with

$$G(x, z) := \varphi^z(x),$$

where $\varphi^z(x) = y(1; x)$ is the solution of the ordinary differential equation

$$\dot{y}(s) = \Phi(y(s))z, \quad y(0) = x, \quad s \in [0, 1].$$

If L is the Lipschitz constant of the matrix function Φ then the Gronwall lemma implies that

$$\|G(x, z) - G(y, z)\| \leq L e^{L\|z\|} \|x - y\| \quad \forall x, y \in D, z \in \mathbb{R}^m.$$

2.3 The main result

For $x \in \mathbb{R}^d$, $U \in \mathfrak{B}(\mathbb{R}^d)$ with $x \notin U$ we denote the set of increments $z \in \mathbb{R}^m$ which send x into U by

$$E^U(x) := \{z \in \mathbb{R}^m : x + G(x, z) \in U\}. \quad (11)$$

We define the following measure assigning for $U \in \mathfrak{B}(\mathbb{R}^d)$

$$Q(U) := \int_{\mathcal{A}} \mu(E^U(y)) P(dy).$$

Remark 2.3 Clearly for

$$\lambda_\varepsilon := \int_{\mathcal{A}} \nu\left(\frac{E^{D^c}(y)}{\varepsilon}\right) P(dy) \quad \text{and} \quad h_\varepsilon := h\left(\frac{1}{\varepsilon}\right), \quad \varepsilon \in (0, 1)$$

equation (7) implies

$$\lim_{\varepsilon \rightarrow 0+} \frac{\lambda_\varepsilon}{h_\varepsilon} = Q(D^c).$$

Theorem 2.1 Let Hypotheses (D.1) and (S.1-2) be fulfilled and suppose that $Q(\partial D) = 0$ and $Q(D^c) > 0$. Then for any $\gamma \in (0, \frac{1}{5})$ any $\theta > 0$ and $U \in \mathfrak{B}(\mathbb{R}^d)$ such that $Q(\partial U) = 0$ the first exit time $\mathbb{T}_y(\varepsilon)$ satisfies

$$\lim_{\varepsilon \rightarrow 0} \sup_{y \in D_{\varepsilon^\gamma}} \left| \mathbb{E} \left[e^{-\theta Q(D^c) h_\varepsilon \mathbb{T}_y(\varepsilon)} \mathbf{1}_{\{X_{\mathbb{T}_y(\varepsilon), y}^\varepsilon \in U\}} \right] - \frac{1}{1 + \theta} \frac{Q(U \cap D^c)}{Q(D^c)} \right| = 0.$$

Corollary 2.1 Under the assumptions of Theorem 2.1 follows

$$Q(D^c) h_\varepsilon \mathbb{T}_x(\varepsilon) \xrightarrow{d} EXP(1),$$

$$\mathbb{P}(X_{\mathbb{T}_x(\varepsilon), x}^\varepsilon \in U) \rightarrow \frac{Q(U \cap D)}{Q(D)}, \quad \varepsilon \rightarrow 0,$$

where the convergence is uniform over all initial values $x \in D_{\varepsilon^\gamma}$.

2.4 Example: Van der Pol oscillator perturbed by α -stable Lévy noise

As a simple but illustrative application of Theorem 2.1 we determine the law of the first exit time of a Van der Pol oscillator perturbed by small Itô-multiplicative α -stable Lévy noise. More precisely, let Z be a bivariate Lévy process with the characteristic function

$$\mathbb{E} \left[e^{i \langle u, Z_t \rangle} \right] = e^{-tc(\alpha) \|u\|^\alpha}, \quad \alpha \in (0, 2), \quad u \in \mathbb{R}^2, \quad c(\alpha) = \frac{\pi}{2^\alpha} \frac{|\Gamma(-\frac{\alpha}{2})|}{\Gamma(1 + \frac{\alpha}{2})},$$

and a Lévy triplet $(0, \nu, 0)$, where

$$\nu(dy) = \mathbf{1}_{\mathbb{R}^2 \setminus \{0\}}(y) \frac{dy}{\|y\|^{2+\alpha}}.$$

Clearly, ν is a regularly varying measure of index $-\alpha$ with the limit measure $\mu = \nu$ and a scaling function

$$h(r) = \int_{\|y\| \geq r} \frac{dy}{\|y\|^{2+\alpha}} = \frac{2\pi}{\alpha} \frac{1}{r^\alpha}.$$

Consider a Van der Pol oscillator for $u = (u_1, u_2)$ and $f = (f_1, f_2)$

$$\dot{u} = f(u), \quad \begin{cases} f_1(u_1, u_2) &= u_1, \\ f_2(u_1, u_2) &= -u_1 + (1 - u_1^2)u_2. \end{cases}$$

which has an unstable stationary solution $u \equiv 0$ and a unique periodic solution $u^\circ = (u_1^\circ(t), u_2^\circ(t))_{t \in [0, T^\circ]}$ of period $T^\circ > 0$ irrespective of initial values which we can omit since all quantities involved will

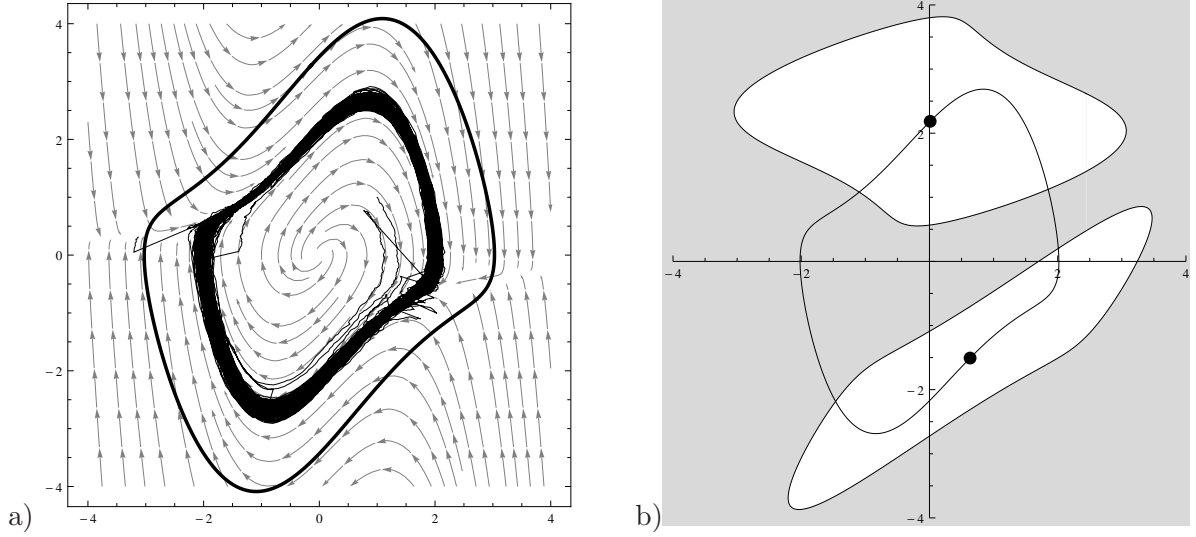


Figure 1: a) A typical exit path of a Van der Pol oscillator perturbed by 1.9-stable Lévy noise; b) the domains $G^{\ominus}(D^c - u^{\circ}(t))$ in the space of noise jumps for two different values of $t \in [0, T^{\circ}]$.

not depend on them. It is well known that the set $\mathcal{A} = \{(u_1^{\circ}(t), u_2^{\circ}(t))_{t \in [0, T^{\circ}]}\} \subset \mathbb{R}^2$ is an exponentially orbitally stable limit cycle. In particular for any bounded and measurable function $\varphi : \mathbb{R}^d \rightarrow (0, \infty) > 0$, any initial point $x \neq 0$, we have

$$\frac{1}{t} \int_0^t \varphi(u(s, x)) ds \rightarrow \frac{1}{T^{\circ}} \int_0^{T^{\circ}} \varphi(u^{\circ}(s)) ds = \int_{\mathcal{A}} \varphi(v) P(dv),$$

$$\text{where } P(B) = \frac{1}{T^{\circ}} \int_0^{T^{\circ}} \mathbf{1}_B(u^{\circ}(s)) ds, \quad \text{for } B \in \mathfrak{B}(\mathbb{R}^2),$$

and this convergence is uniform over all $x \in D$ bounded away from the origin. Consider now a Van der Pol oscillator perturbed by multiplicative Itô noise

$$dX_t^{\varepsilon} = f(X_t^{\varepsilon})dt + \varepsilon G(X_t^{\varepsilon})dZ_t$$

where $(x_1, x_2) \mapsto G(x_1, x_2)$ is a 2×2 matrix valued function satisfying Hypotheses (S.1) and (S.2) of Section 2. Let D be an open bounded invariant domain of attraction containing the limit cycle \mathcal{A} with $\text{dist}(\mathcal{A}, \partial D) > 0$, see Fig. 1. Let

$$G_t = G(u_1^{\circ}(t), u_2^{\circ}(t)) \quad \text{and} \quad G_t^{\ominus} := \begin{cases} G_t^{-1}, & \det G_t \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For any $\delta > 0$ we can choose a small neighbourhood $\mathcal{B}_{\delta}(0)$ of the unstable fixed point 0 of the Van der Pol oscillator, such that the domain $D^{(\delta)} = D \setminus \mathcal{B}_{\delta}(0)$ and f satisfy Hypothesis (D.1). Let $x \in D^{(\delta)}$. Denote by $\mathbb{T}_x^{(\delta)}(\varepsilon)$ the first exit time from the domain $D^{(\delta)}$. We are now in the state to apply Theorem 2.1 and find that

$$\varepsilon^{\alpha} \frac{2\pi}{\alpha T^{\circ}} \int_0^{T^{\circ}} \left[\int_{G_s^{\ominus} D^c} + \int_{G_s^{\ominus} \mathcal{B}_{\delta}(0)} \frac{dy}{\|y - u^{\circ}(s)\|^{2+\alpha}} \right] ds \cdot \mathbb{T}_x^{(\delta)}(\varepsilon) \rightarrow EXP(1), \quad \varepsilon \rightarrow 0.$$

Taking into account that $\int_{\mathcal{B}_{(\delta)}} dy \rightarrow 0$ as $\delta \rightarrow 0$ we finally obtain the limiting law for $\mathbb{T}_y(\varepsilon)$ such that

$$\varepsilon^\alpha \left(\frac{2\pi}{\alpha T^\circ} \int_0^{T^\circ} \int_{G_s^\ominus D^c} \frac{dy}{\|y - u^\circ(s)\|^{2+\alpha}} ds \right) \cdot \mathbb{T}_x^{(0)}(\varepsilon) \xrightarrow{d} EXP(1), \quad \varepsilon \rightarrow 0,$$

with the convergence uniformly over all $x \in D \setminus \mathcal{B}_{\varepsilon^{10}}(\partial D^\times)$ with $D^\times = D \setminus \{0\}$.

3 Small jumps dynamics

The aim of this section is to determine the precise asymptotics of $(X_{t,x}^\varepsilon)_{t \in [0, T_1]}$ for the first large jump times T_1 . This will be accomplished in Proposition 3.1, which tell us that for times $t \in [0, T_1)$ the deterministic dynamics and its ergodicity property dominates, and at $t = T_1$ there occurs a single large jump. We assume Hypotheses **(D.1)** and **(S.1-2)** to be satisfied in the sequel.

3.1 Asymptotics until the first large jump

Let $\rho = \rho_\varepsilon$, $\varepsilon \in (0, 1]$, be a positive sequence, which is monotonically increasing to infinity, $\rho_\varepsilon \nearrow \infty$ as $\varepsilon \searrow 0$ and denote by

$$\beta_\varepsilon := \nu(\mathcal{B}_{\rho^\varepsilon}^c).$$

Consider the following ε -dependent Lévy-Itô decomposition $Z_t := \xi_t^\varepsilon + \eta_t^\varepsilon$ for all $t \geq 0$, $\varepsilon \in (0, 1)$,

$$\begin{aligned} \eta_t^\varepsilon &:= \int_{(0,t]} \int_{\|z\| > \rho^\varepsilon} z N(ds, dz), \\ \xi_t^\varepsilon &:= Z_t - \eta_t^\varepsilon = b_\varepsilon t + A^{\frac{1}{2}} B_t + \int_{(0,t]} \int_{0 < \|z\| \leq \rho^\varepsilon} z \tilde{N}(ds, dz), \\ b_\varepsilon &:= b + \mathbb{E} \left[\int_{(0,1]} \int_{\{1 < \|y\| \leq \rho^\varepsilon\}} y N(ds, dz) \right] = b + \int_{1 < \|z\| \leq \rho^\varepsilon} z \nu(dz). \end{aligned} \tag{12}$$

The compound Poisson process η^ε here is characterised by a family of i.i.d. waiting times $(\tau_i^\varepsilon)_{i \in \mathbb{N}}$ with $\tau_i^\varepsilon \sim \text{EXP}(\beta_\varepsilon)$, the renewal times

$$T_i^\varepsilon = \sum_{k=1}^i \tau_k^\varepsilon,$$

and an family of i.i.d. large jumps $(W_i^\varepsilon)_{i \in \mathbb{N}}$, also independent of $(\tau_i^\varepsilon)_{i \in \mathbb{N}}$ with $W_i^\varepsilon \sim \nu_\varepsilon$, where

$$\nu_\varepsilon(\cdot) = \frac{\nu(\cdot \cap \mathcal{B}_{\rho^\varepsilon}^c)}{\nu(\mathcal{B}_{\rho^\varepsilon}^c)}. \tag{13}$$

The process ξ^ε is a Lévy process with jumps bounded from above by ρ^ε and hence has all finite moments.

3.2 Control of the small jump noise

In this subsection we show that the probabilities of deviations of bounded integrals driven by the small noise $\varepsilon \xi^\varepsilon$ defined in (S.2) decay exponentially.

Lemma 3.1 Let $(\delta_\varepsilon)_{\varepsilon \in (0,1]}$ be a monotone sequence with $\delta_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$ satisfying in addition

$$\lim_{\varepsilon \rightarrow 0+} \varepsilon \frac{\rho^\varepsilon}{\delta_\varepsilon} = 0. \quad (14)$$

Then for any $C > 0$ there is $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$

$$\frac{\varepsilon \|b_\varepsilon\|}{\delta_\varepsilon} \leq C. \quad (15)$$

Proof: In order to prove (15) we center the process ξ , $\tilde{\xi}_t := \xi_t - b_\varepsilon t$, such that $\tilde{\xi}_t$ is a Lévy martingale with jumps bounded from above by ρ^ε . Since $\|b_\varepsilon\| \leq \|b\| + \|\int_{1 < \|z\| \leq \rho^\varepsilon} y \nu(dy)\|$ we obtain by Jensen's inequality and the regular variation of the function h defined by (6) that

$$\left\| \int_{1 < \|z\| \leq \rho^\varepsilon} y \nu(dy) \right\|^2 \leq \int_{1 < \|z\| \leq \rho^\varepsilon} \|y\|^2 \nu(dy) = - \int_1^{\rho^\varepsilon} r^2 h(dr) \leq (\rho^\varepsilon)^2 h(1),$$

such that $\|b_\varepsilon\| \leq \|b\| + \sqrt{h(1)} \rho^\varepsilon$, which gives the desired result with the help of (14). \blacksquare

Lemma 3.2 Let $(\delta_\varepsilon)_{\varepsilon \in (0,1]}$ be a monotone sequence with $\delta_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$ and $p \geq 1$ satisfying

$$\lim_{\varepsilon \rightarrow 0+} \frac{\varepsilon \rho^\varepsilon}{\delta_\varepsilon^{(p+1)/2}} = 0. \quad (16)$$

Then for all $T > 0$ and $C > 0$ there is $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$

$$\mathbb{P}([\varepsilon \tilde{\xi}]_{\tau^\varepsilon} > C \delta_\varepsilon^p) \leq e^{-C \delta_\varepsilon^{-1} + 1}.$$

Proof: The discontinuous part of the quadratic variation process $[\varepsilon \tilde{\xi}]_t^d = [\varepsilon \tilde{\xi}]_t - \text{trace}(A) \varepsilon^2 t$ is a Lévy subordinator and has the representation

$$[\varepsilon \tilde{\xi}]_t^d = \varepsilon^2 \sum_{s \leq t} \|\Delta \tilde{\xi}_s\|^2 = \varepsilon^2 \int_{(0,t]} \int_{0 < \|z\| \leq \rho^\varepsilon} \|z\|^2 N(dz, ds) \quad t \geq 0 \text{ a.s.}$$

Since the jumps of $[\varepsilon \tilde{\xi}]_t^d$ by construction are bounded by $(\varepsilon \rho^\varepsilon)^2 \leq 1$, its Laplace transform is well-defined for all $\lambda \in \mathbb{R}$ and $t \geq 0$

$$\begin{aligned} \mathbb{E} \left[e^{\lambda [\varepsilon \tilde{\xi}]_t^d} \right] &= \exp \left(t \int_{0 < \|y\| \leq \rho^\varepsilon} (e^{\lambda^2 \varepsilon^2 \|y\|^2} - 1) \nu(dy) \right) \\ &= \exp \left(-t \int_{0 < r \leq \rho^\varepsilon} (e^{\lambda^2 \varepsilon^2 r^2} - 1) h(dr) \right). \end{aligned}$$

For any $\lambda > 0$ the exponential Chebyshev inequality yields

$$\begin{aligned} \mathbb{P}([\varepsilon \tilde{\xi}]_T^d > C \delta_\varepsilon^p) &\leq \mathbb{P}(e^{\lambda [\varepsilon \tilde{\xi}]_T^d} > e^{\lambda C \delta_\varepsilon^p}) \leq e^{-\lambda C \delta_\varepsilon^p} \mathbb{E}[e^{\lambda [\varepsilon \tilde{\xi}]_T^d}] \\ &= \exp \left(-\lambda C \delta_\varepsilon^p - T \int_{0 < r \leq \rho^\varepsilon} (e^{\lambda^2 \varepsilon^2 r^2} - 1) h(dr) \right). \end{aligned}$$

We continue with the help of $e^s - 1 \leq 2s$ for small s . Replacing λ by $\delta_\varepsilon^{-(p+1)}$ we ensure the smallness of the argument noting that by (16) $\sup_{0 < r \leq \rho^\varepsilon} \varepsilon^2 r^2 / \delta_\varepsilon^{p+1} \leq (\varepsilon \rho^\varepsilon)^2 / \delta_\varepsilon^{p+1} \rightarrow 0$ for $\varepsilon \rightarrow 0+$. We obtain

$$\left| T \int_{0 < r \leq \rho^\varepsilon} (e^{\varepsilon^2 r^2 / \delta_\varepsilon^{p+1}} - 1) h(dr) \right|$$

$$\begin{aligned}
&\leq |2T\varepsilon^2/\delta_\varepsilon^{p+1}(\int_{0 < r \leq 1} + \int_{1 < r \leq \rho^\varepsilon})r^2h(dr)| \\
&\leq 2T\varepsilon^2/\delta_\varepsilon^{p+1}|\int_{0 < r \leq 1} r^2h(dr)| + 2T(\varepsilon\rho^\varepsilon)^2/\delta_\varepsilon^{p+1}|\int_{1 < r \leq \rho^\varepsilon} h(dr)| \\
&\leq cT(\varepsilon\rho^\varepsilon)^2/\delta_\varepsilon^{p+1}.
\end{aligned}$$

Therefore by (16) there is $\varepsilon_0 \in (0, 1)$ such that $\varepsilon \in (0, \varepsilon_0]$ implies the final result

$$\mathbb{P}([\varepsilon\tilde{\xi}]_T > C\delta_\varepsilon^p) \leq \exp(-C\delta_\varepsilon^{-1} + \text{trace}(A)\varepsilon^2T + cT(\varepsilon\rho^\varepsilon)^2/\delta_\varepsilon^{p+1}) \leq \exp(-C\delta_\varepsilon^{-1} + 1).$$

■

In the following lemma we estimate the deviation of the stochastic integral with respect to the (local) martingale part $\tilde{\xi}^\varepsilon$ of the small jumps noise process ξ^ε

$$\tilde{\xi}_t^\varepsilon = A^{1/2}B_t + \int_{0 < \|y\| \leq \rho^\varepsilon} y\tilde{N}(t, dy).$$

Lemma 3.3 *Let $(g_t)_{t \geq 0}$ be an adapted, càdlàg process with bounded values by C_g in $\mathbb{R}^{m \otimes d}$ for a suitable matrix norm. For all $T > 0$ and functions δ_ε and ρ^ε satisfying (16) for $p = 4$ there is $\varepsilon_0 \in (0, 1)$ and a constant $C_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$*

$$\mathbb{P}(\sup_{s \in [0, T]} \varepsilon \sum_{i=1}^d \left| \sum_{j=1}^m \int_0^t g_{s-}^{ij} d\tilde{\xi}^j(s) \right| > \delta_\varepsilon) \leq \exp(-C_0\delta_\varepsilon^{-1} + \ln(6d)).$$

Proof: Suppose $\max_{i,j} \sup_{t \geq 0} |g_t^{ij}| \leq C_g$ almost surely. We consider the each component of the d -dimensional martingale

$$M_t^i = \sum_{j=1}^m \int_0^t g_{s-}^{ij} d\tilde{\xi}^j(s).$$

By construction $\|\Delta_t M\| \leq m d C_g \rho^\varepsilon =: C\rho^\varepsilon$ almost surely. We estimate the probability of a deviation of size δ_ε from zero conditioned on small quadratic variation

$$\mathbb{P}(\sup_{s \in [0, T]} \|\varepsilon M_s\| > \delta_\varepsilon) \leq \mathbb{P}(\sup_{s \in [0, T]} \|\varepsilon M_s\| > \delta_\varepsilon \mid [\varepsilon M]_T \leq \delta_\varepsilon^4) + \mathbb{P}([\varepsilon M]_T > \delta_\varepsilon^4). \quad (17)$$

Step 1: We estimate the first term of inequality (17). Following the lines of the proofs of Lemma 26.19 and Theorem 26.17 part (i) in [20] we find the following estimate. For any $\lambda > 0$

$$\mathbb{P}(\sup_{s \in [0, T]} \varepsilon M_s^i > \delta_\varepsilon \mid [\varepsilon M]_T \leq \delta_\varepsilon^4) \leq \exp(-\lambda\delta_\varepsilon + \lambda^2 \Upsilon(\lambda C_g \varepsilon \rho^\varepsilon) \delta_\varepsilon^4),$$

where $\Upsilon : (0, \infty) \rightarrow (0, \infty)$, $\Upsilon(x) = -(x + \ln(1-x))x^{-2}$. Replacing λ by $\lambda_\varepsilon = \delta_\varepsilon^{-2}$ and keeping in mind that $\lim_{\varepsilon \rightarrow 0+} \Upsilon(\lambda C_g \varepsilon \rho^\varepsilon) = \frac{1}{2}$ yields

$$\mathbb{P}(\sup_{s \in [0, T]} \varepsilon M_s^i > \delta_\varepsilon \mid [\varepsilon M]_T \leq \delta_\varepsilon^4) \leq \exp(-\delta_\varepsilon^{-1}).$$

For the infimum of the negative analogue holds the respective estimate, which provides for each i for $\lambda_\varepsilon = d\delta_\varepsilon^{-2}$ instead

$$\mathbb{P}(\sup_{s \in [0, T]} |\varepsilon M_s^i| > \frac{\delta_\varepsilon}{d} \mid [\varepsilon M]_T \leq \delta_\varepsilon^4) \leq \exp(-\delta_\varepsilon^{-1} + \ln(2)),$$

where the right-hand side does not depend on i , such that eventually

$$\begin{aligned}
& \mathbb{P}(\sup_{s \in [0, T]} \|\varepsilon M_s\| > \delta_\varepsilon \mid [\varepsilon M]_T \leq \delta_\varepsilon^4) \\
& \leq \sum_{i=1}^d \mathbb{P}(\sup_{s \in [0, T]} |\varepsilon M_s^i| > \frac{\delta_\varepsilon}{m} \mid [\varepsilon M]_T \leq \delta_\varepsilon^4) \\
& \leq \exp(-\delta_\varepsilon^{-1} + \ln(2d)).
\end{aligned}$$

Step 2: We treat the second term in inequality (17). The boundedness assumption of g yields

$$[\varepsilon M]_t = \int_0^t \|g_{s-}^* - g_{s-}\|^2 d[\varepsilon A^{\frac{1}{2}} B]_s + \int_0^t \|g_{s-}^* - g_{s-}\|^2 d[\varepsilon \tilde{\xi}]_s^d \leq C^2(\varepsilon^2 \text{trace}(A)t + [\varepsilon \tilde{\xi}]_t^d), \quad t \geq 0.$$

Hence

$$\mathbb{P}([\varepsilon M]_T \geq \delta_\varepsilon^4) \leq \mathbb{P}(C^2[\varepsilon \tilde{\xi}]_T^d \geq \frac{1}{2}\delta_\varepsilon^4) + \mathbb{P}(C^2 \text{trace}(A)\varepsilon^2 T \geq \frac{1}{2}\delta_\varepsilon^4).$$

The second term vanishes by (16), which implies $\varepsilon^2 < \delta_\varepsilon^4$ for small $\varepsilon \in (0, 1)$. The first term is treated as in Lemma 3.2. Eventually

$$\mathbb{P}([\varepsilon M]_T \geq \delta_\varepsilon^4) \leq \mathbb{P}([\varepsilon \tilde{\xi}]_T^d \geq \frac{1}{2C^2}\delta_\varepsilon^4) \leq \exp(-\frac{\delta_\varepsilon^{-1}}{2C^2} + 1).$$

Combining Step 1 and 2 yields a constant $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$

$$\mathbb{P}(\sup_{s \in [0, T]} \|\varepsilon M_s\| > \delta_\varepsilon) \leq \exp(-\min(1, \frac{1}{2C^2})\delta_\varepsilon^{-1} + \ln(2de)).$$

This finishes the proof. ■

3.3 Localization of V^ε close to u up to a fixed time

Let V^ε be the solution of equation (8), where the driving noise Z is replaced by the ε -dependent small jumps part ξ^ε of Z as defined in (12). The first large jump time $T_1 > 0$ is exponentially distributed by with intensity $\beta_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$. By definition then

$$V_{t,x}^\varepsilon = X_{t,x}^\varepsilon \quad \text{for } t \in [0, T_1).$$

In order to study the fluctuations of $X_{t,x}^\varepsilon$ for $t < T_1$ we introduce

$$\mathbb{T}_x^*(\varepsilon) := \inf\{t > 0 \mid V_{t,x}^\varepsilon \notin D\}.$$

Lemma 3.4 (Non-exit up to fixed times) *For any $T \geq 0$ there is $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and δ_ε satisfying (16) there*

$$\sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathbb{T}_x^*(\varepsilon) \leq T) \leq \exp(-\delta_\varepsilon^{-1} + 2 + \ln(d)).$$

Proof: By Remark 2.2 for any sufficiently small δ_ε and $x \in D_{\delta_\varepsilon}$ follows

$$\text{dist}(u(t; x), \partial D) \geq \delta_\varepsilon \quad \forall t \geq 0.$$

Since $\mathbb{T}_x^*(\varepsilon)$ denotes the exit from D , we infer

$$\{\mathbb{T}_x^*(\varepsilon) \leq T\} = \{\mathbb{T}_x^*(\varepsilon) \leq T\} \cap \left\{ \sup_{t \in [0, \mathbb{T}_x^*(\varepsilon)]} \|V_{t,x} - u(t; x)\| > \delta_\varepsilon \right\}.$$

We lighten notation $V = V^\varepsilon$, $\mathbb{T}^* = \mathbb{T}_x^+(\varepsilon)$ etc. Then for $t \leq T$ follows by definition

$$\begin{aligned} & V_{t \wedge \mathbb{T}^*, x} - u(t \wedge \mathbb{T}^*; x) \\ &= \int_0^{t \wedge \mathbb{T}^*} f(V_{s,x}) - f(u(s; x)) ds + \varepsilon \int_0^{t \wedge \mathbb{T}^*} H(V_{s,x}) b_\varepsilon ds + \varepsilon \int_0^{t \wedge \mathbb{T}^*} F(V_{s,x}) dA^{\frac{1}{2}} B_s \\ & \quad + \int_0^{t \wedge \mathbb{T}^*} \int_{0 < \|z\| \leq \rho^\varepsilon} G(V_{s-,x}, \varepsilon z) \tilde{N}(ds, dz). \end{aligned} \quad (18)$$

We fix the constant

$$C_D := \sup_{\substack{v \in D \\ w \in B_1}} \max\{L, \|f(v)\|, \|H(v)\|, \|F(v)\|, \|G(v, w)\|\}. \quad (19)$$

The global Lipschitz property of f on D and the standard integral version of Gronwall's lemma yield

$$\begin{aligned} & \sup_{x \in D_{\delta_\varepsilon}} \sup_{t \in [0, T \wedge \mathbb{T}^*]} \|V_{t,x} - u(t; x)\| \\ & \leq e^{C_D T} \sup_{x \in D_{\delta_\varepsilon}} \sup_{t \in [0, T \wedge \mathbb{T}^*]} \left\| \varepsilon \int_0^t H(V_{s,x}) b_\varepsilon ds + \varepsilon \int_0^t F(V_{s,x}) dA^{\frac{1}{2}} B_s \right. \\ & \quad \left. + \int_0^t \int_{0 < \|z\| \leq \rho^\varepsilon} G(V_{s-,x}, \varepsilon z) \tilde{N}(ds, dz) \right\|. \end{aligned} \quad (20)$$

The representation (18) has the (local) martingale part

$$M_{t,x} := \varepsilon \int_0^t F(V_{s,x}) d(A^{\frac{1}{2}} B_s) + \int_0^t \int_{0 < \|z\| \leq \rho^\varepsilon} G(V_{s-,x}, \varepsilon z) \tilde{N}(ds, dz). \quad (21)$$

The previous lemma yields for i -th component $M_{t,x}^i$ and any $\lambda > 0$

$$\begin{aligned} & \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathbb{T}_x^*(\varepsilon) \leq T) \\ &= \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathbb{T}_x^*(\varepsilon) \leq T, \sup_{t \in [0, T \wedge \mathbb{T}^*]} \|V_{t,x} - u(t; x)\| > \delta_\varepsilon) \\ &\leq \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathbb{T}_x^*(\varepsilon) \leq T, \sup_{t \in [0, T \wedge \mathbb{T}^*]} e^{C_D T} \varepsilon \left\| \int_0^t H(V_{s,x}) b_\varepsilon ds \right\| > \frac{\delta_\varepsilon}{2}) \\ & \quad + \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathbb{T}_x^*(\varepsilon) \leq T, \sup_{t \in [0, T \wedge \mathbb{T}^*]} e^{C_D T} \|M_{t,x}\| > \frac{\delta_\varepsilon}{2} \mid [\varepsilon \xi]_T \leq \delta_\varepsilon^4) \\ & \quad + \mathbb{P}([\varepsilon \xi]_T > \delta_\varepsilon^4) \\ &\leq \mathbb{P}(\varepsilon \|b_\varepsilon\| e^{C_D T} T C_D > \frac{\delta_\varepsilon}{2}) + \mathbb{P}([\varepsilon \xi]_T > \delta_\varepsilon^4) \\ & \quad + \sum_{i=1}^d \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\sup_{t \in [0, T \wedge \mathbb{T}^*]} M_{t,x}^i > \frac{\delta_\varepsilon}{2d} \mid [\varepsilon \xi]_T \leq \delta_\varepsilon^4) + \sup_{x \in D_{2\delta_\varepsilon}} \mathbb{P}(\sup_{t \in [0, T \wedge \mathbb{T}^*]} M_{t,x}^i < -\frac{\delta_\varepsilon}{2d} \mid [\varepsilon \xi]_T \leq \delta_\varepsilon^4) \end{aligned}$$

$$\leq \exp(-\delta_\varepsilon^{-1} + 1) + 2d \exp\left(-\lambda \frac{\delta_\varepsilon}{2d} + \lambda^2 \Upsilon(C_D \lambda) \delta_\varepsilon^2\right). \quad (22)$$

The vanishing of the formal first term in the third to last line is the direct consequence of Lemma 3.1. We note that the last inequality is valid for any local martingale with jumps bounded from above by C_D . This is satisfied since by (14) $\lim_{\varepsilon \rightarrow 0+} \varepsilon \rho^\varepsilon = 0$ and for $x \in D$ and $s \in [0, \mathbb{T}_x^*]$

$$\|\Delta_s V_{\cdot, x}^\varepsilon\| \leq \sup_{\substack{v \in \mathcal{B}_D \\ w \in \mathcal{B}_{\varepsilon \rho^\varepsilon}}} \|G(v, w)\| \leq C_D,$$

where the last inequality stems from **(S.2)** part 4. We may now replace in inequality (22) λ by $2d/\delta_\varepsilon^2$ and exploit that $\lim_{r \rightarrow \infty} \Upsilon(r) = \frac{1}{2}$. This yields the desired estimate and finishes the proof. \blacksquare

Corollary 3.1 (Localization up to a fixed time T) *For all $T > 0$ there is $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and δ_ε satisfying (16) follows*

$$\sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}\left(\sup_{s \in [0, T]} \|V_{s, x}^\varepsilon - u(s; x)\| > \delta_\varepsilon\right) \leq \exp(-\delta_\varepsilon^{-1} + 3 + \ln(d)).$$

Proof: On the event $\{\mathbb{T}_x^* > T\}$ we repeat (18), (20) and (22) replacing $t \wedge \mathbb{T}_x^*$ by $t \in [0, T]$. This directly yields the desired result. \blacksquare

3.4 Localization and ergodicity of V^ε

Lemma 3.5 (Non-exit) *For functions ρ^ε , δ_ε and β_ε satisfying the relation (16) there exist constants $C > 0$ and $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$*

$$\sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\exists t \in [0, T_1] : V_{t, x}^\varepsilon \notin D) \leq \frac{C e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2}.$$

Proof: Due to the independence of T_1 and V^ε we calculate

$$\mathbb{P}(\exists t \in [0, T_1] : V_{t, x}^\varepsilon \notin D) \leq \mathbb{P}(\exists t \in [0, \frac{1}{\beta_\varepsilon \delta_\varepsilon}] : V_{t, x}^\varepsilon \notin D) + \mathbb{P}(T_1 > \frac{1}{\beta_\varepsilon \delta_\varepsilon})$$

By construction

$$\mathbb{P}(T_1 > \frac{1}{\beta_\varepsilon \delta_\varepsilon}) = e^{-\delta_\varepsilon^{-1}} \rightarrow 0.$$

Recall by Remark 2.1 that $t \geq \mathcal{S}$ and $x \in D$ imply $u(t; x) \in \mathcal{I}$. Hence

$$\begin{aligned} \mathbb{P}(\exists t \in [0, \frac{1}{\beta_\varepsilon \delta_\varepsilon}] : V_{t, x}^\varepsilon \notin D) &= \int_0^{(\beta_\varepsilon \delta_\varepsilon)^{-1}} \beta_\varepsilon e^{-\beta_\varepsilon s} \mathbb{P}(\exists t \in [0, s] : V_{t, x}^\varepsilon \notin D) ds \\ &\leq \sum_{k=1}^{\lceil (\beta_\varepsilon \delta_\varepsilon \mathcal{S})^{-1} \rceil} \int_{(k-1)\mathcal{S}}^{k\mathcal{S}} \beta_\varepsilon e^{-\beta_\varepsilon s} \mathbb{P}(\exists t \in [0, s] : V_{t, x}^\varepsilon \notin D) ds \\ &\leq \sum_{k=1}^{\lceil (\beta_\varepsilon \delta_\varepsilon \mathcal{S})^{-1} \rceil} \mathbb{P}(\exists t \in [0, k\mathcal{S}] : V_{t, x}^\varepsilon \notin D) e^{-\beta_\varepsilon \mathcal{S} k}. \end{aligned}$$

We denote by

$$\mathcal{E}_x(\varepsilon) := \left\{ \sup_{t \in [0, T]} \|V_{t, x} - u(t; x)\| \leq \delta_\varepsilon \right\}.$$

For the case $k = 1$, $x \in D_{\delta_\varepsilon}$ Corollary 3.1 yields

$$\mathbb{P}(\mathbb{T}_x^* \in [0, \mathcal{S}]) = \mathbb{P}(\exists t \in [0, \mathcal{S}] : V_{t,x}^\varepsilon \notin D) \leq \mathbb{P}(\sup_{t \in [0, \mathcal{S}]} \|V_{t,x}^\varepsilon - u(t; x)\| > \delta_\varepsilon) \leq C e^{-\delta_\varepsilon^{-1}}.$$

Furthermore, Remark 2.1 states that

$$V_{\mathcal{S},x}^\varepsilon = V_{\mathcal{S},x}^\varepsilon - u(\mathcal{S}; x) + u(\mathcal{S}; x) \in \mathcal{B}_{\delta_\varepsilon}(0) + \mathcal{I} \subset D_{2\delta_\varepsilon}.$$

Exploiting the Markov property at time \mathcal{S} we obtain

$$\begin{aligned} & \mathbb{P}(\mathbb{T}_x^* \in ((k-1)\mathcal{S}, k\mathcal{S}]) \\ &= \mathbb{P}(\{\forall t \in [0, (k-1)\mathcal{S}] : V_{t,x}^\varepsilon \in D\} \cap \{\exists t \in [(k-1)\mathcal{S}, k\mathcal{S}] : V_{t,x}^\varepsilon \notin D\}) \\ &= \mathbb{P}(\{\forall t \in [0, (k-1)\mathcal{S}] : V_{t,x}^\varepsilon \in D\} \cap \{\exists t \in [(k-1)\mathcal{S}, k\mathcal{S}] : V_{t,x}^\varepsilon \notin D\} \cap \mathcal{E}_x) + \mathbb{P}(\mathcal{E}_x^c) \\ &\leq \sup_{x \in D_{2\delta_\varepsilon}} \mathbb{P}(\mathbb{T}_x^*(\varepsilon) \in [(k-2)\mathcal{S}, (k-1)\mathcal{S}]) + C e^{-\delta_\varepsilon^{-1}}. \end{aligned}$$

Therefore a recursive argument leads to

$$\mathbb{P}(\mathbb{T}_x^* \in ((k-1)\mathcal{S}, k\mathcal{S}]) \leq k C e^{-\delta_\varepsilon^{-1}}.$$

Finally summing up we obtain the desired result

$$\mathbb{P}(\exists t \in [0, \frac{1}{\beta_\varepsilon \delta_\varepsilon}] : V_{t,x}^\varepsilon \notin D) \leq \sum_{k=1}^{\lceil (\beta_\varepsilon \delta_\varepsilon \mathcal{S})^{-1} \rceil} k C e^{-\delta_\varepsilon^{-1}} \leq \frac{C e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon \mathcal{S})^2}.$$

■

The proof further yields directly that at the time of the first large jump T_1 the small noise solution V^ε is not far from \mathcal{I} .

Corollary 3.2 *Let the assumptions of Lemma 3.5 be fulfilled. Then for all $\kappa > 0$ there is $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$*

$$\sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(V_{T_1,x}^\varepsilon \in \mathcal{B}_\kappa(\mathcal{I})) \leq \frac{C e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2}. \quad (23)$$

We can now state and prove the main result of this section concerning the behavior of $X_{t \in [0, T_1]}^\varepsilon$.

Proposition 3.1 (Ergodicity including the first large jump) *Let the functions $\rho_\varepsilon, \delta_\varepsilon, \beta_\varepsilon$ satisfy (16) for $p = 4$ and*

$$\lim_{\varepsilon \rightarrow 0+} \frac{e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2} = 0. \quad (24)$$

Consider a set $U \in \mathfrak{B}(\mathbb{R}^d)$ such that

$$\lim_{t \rightarrow \infty} \sup_{x \in D} \frac{1}{t} \int_0^t \mu(E^{\partial U}(u(s; x))) ds = 0. \quad (25)$$

Further, we consider a family $U^\varepsilon \in \mathfrak{B}(\mathbb{R}^d)$ such that for all $\kappa > 0$ there exists $\varepsilon_0 \in (0, 1)$ satisfying for $\varepsilon \in (0, \varepsilon_0]$ that $U^\varepsilon \triangle U \subset \mathcal{B}_\kappa(\partial U)$. Then

$$\lim_{\varepsilon \rightarrow 0+} \sup_{x \in D_{\delta_\varepsilon}} |\mathbb{E} [e^{-T_1 \lambda_\varepsilon} \mathbf{1}\{V_{T_1,x}^\varepsilon + G(V_{T_1,x}^\varepsilon, \varepsilon W) \in U^\varepsilon\}] - \int_{\mathcal{A}} \mathbb{P}(v + G(v, \varepsilon W) \in U) P(dv)]| = 0. \quad (26)$$

Proof: Let $\theta \in (0, 1)$. Then by Hypothesis (D.2) there is $\mathcal{T} = \mathcal{T}_\theta > 0$ such that for all $t \geq \mathcal{T}$

$$\sup_{x \in D} \left| \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \varphi(u(s; x)) ds - \int_{\mathcal{A}} \varphi(v) P(dv) \right| \leq \frac{\theta}{2}. \quad (27)$$

In addition, we choose $\mathcal{T} > \mathcal{S}$. Furthermore there exists $\kappa > 0$ such that

$$\sup_{x \in D} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \mu(E^{\mathcal{B}_\kappa(\partial U)}(u(s; x))) ds \leq \frac{\theta}{2}.$$

Once again we lighten notation $V = V^\varepsilon$. Due to the independence of T_1 and V we may continue for $x \in D_{\delta_\varepsilon}$

$$\begin{aligned} & \mathbb{E}[e^{-T_1 \lambda_\varepsilon} \mathbf{1}\{V_{T_1, x} + G(V_{T_1, x}, \varepsilon W) \in U^\varepsilon\}] \\ & \leq \mathbb{E}[e^{-T_1 \lambda_\varepsilon} \mathbf{1}\{V_{T_1, x} + G(V_{T_1, x}, \varepsilon W) \in U^\varepsilon\}] \\ & \leq \sum_{k=0}^{\infty} \mathbb{E} \int_{k\mathcal{T}}^{(k+1)\mathcal{T}} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)s} \mathbf{1}\{V_{s, x} + G(V_{s, x}, \varepsilon W) \in U^\varepsilon\} ds]. \end{aligned}$$

We define

$$\mathcal{E}_x(\varepsilon) = \left\{ \sup_{t \in [0, \mathcal{T}]} \|V_{t, x} - u(t; x)\| \leq \delta_\varepsilon \right\}$$

and calculate

$$\begin{aligned} & \mathbb{E} \left[\int_{k\mathcal{T}}^{(k+1)\mathcal{T}} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)s} \mathbf{1}\{V_{s, x} + G(V_{s, x}, \varepsilon W) \in U^\varepsilon\} ds \right] \\ & \leq \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)k\mathcal{T}} \mathbb{E} \left[\int_{k\mathcal{T}}^{(k+1)\mathcal{T}} \mathbf{1}\{V_{s, x} + G(V_{s, x}, \varepsilon W) \in U^\varepsilon\} ds \right] \\ & \leq \mathcal{T} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)k\mathcal{T}} \left(\mathbb{E} \left[\frac{1}{\mathcal{T}} \int_{k\mathcal{T}}^{(k+1)\mathcal{T}} \mathbf{1}\{V_{s, x} + G(V_{s, x}, \varepsilon W) \in U^\varepsilon\} \mathbf{1}(\mathcal{E}_x) ds \right] + \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathcal{E}_x^c) \right) \\ & = \mathcal{T} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)k\mathcal{T}} \left(\mathbb{E} \left[\mathbb{E} \left[\frac{1}{\mathcal{T}} \int_{k\mathcal{T}}^{(k+1)\mathcal{T}} \mathbf{1}\{V_{s, x} + G(V_{s, x}, \varepsilon W) \in U^\varepsilon\} \mathbf{1}(\mathcal{E}_x) ds \mid \mathcal{F}_\mathcal{T} \right] \right] + \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathcal{E}_x^c) \right) \\ & \leq \mathcal{T} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)k\mathcal{T}} \left(\sup_{y \in \mathcal{B}_{\delta_\varepsilon}(\mathcal{I})} \mathbb{E} \left[\frac{1}{\mathcal{T}} \int_{(k-1)\mathcal{T}}^{k\mathcal{T}} \mathbf{1}\{V_{s, y} + G(V_{s, y}, \varepsilon W) \in U^\varepsilon\} \right] + \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(\mathcal{E}_x^c) \right). \quad (28) \end{aligned}$$

A recursive argument yields

$$\begin{aligned} & \mathbb{E} \left[\int_{k\mathcal{T}}^{(k+1)\mathcal{T}} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)s} \mathbf{1}\{V_{s, x} + G(V_{s, x}, \varepsilon W) \in U^\varepsilon\} ds \right] \\ & \leq \mathcal{T} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)k\mathcal{T}} \left(\sup_{y \in D_{\delta_\varepsilon}} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \mathbb{P}(V_{s, y} + G(V_{s, y}, \varepsilon W) \in U^\varepsilon) ds \right. \\ & \quad \left. + (k+1) \sup_{y \in D_{\delta_\varepsilon}} \mathbb{P}(\mathcal{E}_y^c) \right) = J. \end{aligned}$$

We choose $\varepsilon_0 \in (0, 1)$ small enough such that $\varepsilon \in (0, \varepsilon_0]$ implies

$$(U^\varepsilon \triangle U) + \mathcal{B}_{(1+Le^{L^2})\delta_\varepsilon}(0) \subset \mathcal{B}_\kappa(\partial U).$$

Hence we may continue

$$\begin{aligned} J &\leq \mathcal{T} \beta_\varepsilon e^{-(\beta_\varepsilon + \lambda_\varepsilon)k\mathcal{T}} \left(\sup_{y \in D_{\delta_\varepsilon}} \left(\frac{1}{\mathcal{T}} \int_0^\mathcal{T} \mathbb{P}(u(s; y) + G(u(s; y), \varepsilon W) \in U^\varepsilon + \mathcal{B}_{(1+Le^{L^2})\delta_\varepsilon}) ds \right. \right. \\ &\quad \left. \left. + (k+1) \exp(-\delta_\varepsilon^{-1} + 3 + \ln(d)) \right) \right). \end{aligned}$$

The first summand in the brackets satisfies due to the regular variation of ν , the measure continuity and conditions (25) and (27)

$$\begin{aligned} &\frac{\beta_\varepsilon}{h_\varepsilon} \frac{1}{\mathcal{T}} \int_0^\mathcal{T} \mathbb{P}(u(s; x) + G(u(s; x), \varepsilon W) \in \mathcal{B}_\kappa(U^\varepsilon)) \triangle U ds \\ &\leq \frac{1}{\mathcal{T}} \int_0^\mathcal{T} \frac{1}{h_\varepsilon} \nu \left(\frac{1}{\varepsilon} E^{\mathcal{B}_\kappa(\partial U)}(u(s; x)) \right) ds \\ &\leq (1+\theta) \frac{1}{\mathcal{T}} \int_0^\mathcal{T} \mu \left(E^{\mathcal{B}_\kappa(\partial U)}(u(s; x)) \right) ds \leq (1+\theta) \frac{\theta}{2}. \end{aligned}$$

Hence

$$\begin{aligned} &\sup_{y \in D_{\delta_\varepsilon}} \left(\frac{1}{\mathcal{T}} \int_0^\mathcal{T} \mathbb{P}(u(s; y) + G(u(s; y), \varepsilon W) \in \mathcal{B}_\kappa(U^\varepsilon)) ds \right. \\ &\quad \left. \leq (1+\theta) \frac{1}{\mathcal{T}} \int_0^\mathcal{T} \mu \left(E^U(u(s; x)) \right) ds + (1+\theta) \frac{\theta}{2} \frac{h_\varepsilon}{\beta_\varepsilon} \right) \end{aligned}$$

We eventually obtain

$$\frac{1}{\mathcal{T}} \int_0^\mathcal{T} \mu \left(E^U(u(s; x)) \right) ds \leq (1+\theta) \int_{\mathcal{A}} \mu \left(E^U(v) \right) P(dv).$$

Summing up over k we end up with an $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$\begin{aligned} J &\leq (1+\theta)^2 \frac{\mathcal{T} \beta_\varepsilon}{1 - e^{-\beta_\varepsilon \mathcal{T}}} \left(\int_{\mathcal{A}} \mu \left(E^U(v) \right) P(dv) + (1+\theta) \frac{\theta}{2} \frac{h_\varepsilon}{\beta_\varepsilon} \right) + \frac{\beta_\varepsilon}{(1 - e^{-\beta_\varepsilon \mathcal{T}})^2} \exp(-\delta_\varepsilon^{-1} + 3 + \ln(d)) \\ &\leq (1+\theta)^3 \left(\int_{\mathcal{A}} \mu \left(E^U(v) \right) P(dv) + \frac{\theta}{2} \right) \end{aligned}$$

This closes the proof. ■

4 Proof of the Theorem 2.1

In this section we exploit the results on $(X_{t,x}^\varepsilon)_{t \in [0, T_1]}$ and the strong Markov property to pass from $[0, T_1]$ to $[T_{k-1}, T_k]$ in order to determine the first exit scenario of $(X_{t,x}^\varepsilon)_{t \geq 0}$. The main step consists in the upper bound of the Laplace transform.

4.1 The upper bound

Proposition 4.1 *Assume Hypotheses (D.1) and (S.1-2) to be satisfied. We choose $\delta_\varepsilon = \varepsilon^\gamma$ for $\gamma > 0$ and $\rho^\varepsilon = \varepsilon^{-\rho}$ for $\rho \in (0, 1)$ such that conditions (16) for $p = 4$ and (24) are satisfied. Furthermore we assume that*

$$\int_{\mathcal{A}} \mu(E^{\partial D}(y)) P(dy) = 0 \quad \text{and} \quad \int_{\mathcal{A}} \mu(E^{D^c}(y)) P(dy) > 0. \quad (29)$$

Then for all $\theta > 0$ and $U \in \mathfrak{B}(\mathbb{R}^d)$ such that

$$\int_{\mathcal{A}} \mu(E^{\partial U}(y)) P(dy) = 0 \quad (30)$$

and $C \in (0, 1)$ there is $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$ the first exit time $\mathbb{T}_y = \mathbb{T}_y(\varepsilon)$ satisfies

$$\sup_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q(D^c) h_\varepsilon \mathbb{T}_y} \mathbf{1}\{X_{\mathbb{T}_y, y}^\varepsilon \in U\} \right] \leq (1 + C) \frac{1}{1 + \theta} \frac{Q(U \cap D^c)}{Q(D^c)}.$$

Proof: We start by lightening the notation. Whenever we consider the first jump $i = 1$ we omit the index. Hence we write $T = T_1 = T_1^\varepsilon$, $W = W_1 = W_1^\varepsilon$ etc. Define $\tau_i := T_i - T_{i-1}$. All processes will loose their ε index. For convenience we abbreviate $Q = Q(D^c)$. We define the following events for $y \in D_{\delta_\varepsilon}$ and $s, t \geq 0$ by

$$\begin{aligned} A_{t,s,y} &:= \{X_{r,\cdot} \circ \theta_s(y) \in D \text{ for all } r \in [0, t]\}, \\ B_{t,s,y} &:= \{X_{r,\cdot} \circ \theta_s(y) \in D \text{ for all } r \in [0, t), X_{t,\cdot} \circ \theta_s(y) \notin D\} \\ O_{t,s,y}(U) &:= \{X_{t,\cdot} \circ \theta_s(y) \in U\}. \end{aligned}$$

For $x \in D_{\delta_\varepsilon}$ and with the convention $T_0 = 0$ we denote the trivial disjoint repartition

$$\{\mathbb{T}_x < \infty\} = \bigcup_{k=1}^{\infty} \{\mathbb{T}_x \in (T_{k-1}, T_k)\} \cup \{\mathbb{T}_x = T_k\}.$$

Furthermore consider for $k \geq 1$ and

$$\{\mathbb{T}_x = T_k\} = \bigcap_{i=1}^{k-1} A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}} \cap B_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}$$

and analogously

$$\{\mathbb{T}_x \in (T_{k-1}, T_k)\} = \bigcap_{i=1}^{k-1} A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}} \cap \{V_t^k \circ \theta_{T_{k-1}}(x) \notin D \text{ for some } t \in (0, \tau_k)\}.$$

Therefore we may calculate

$$\mathbf{1}\{\mathbb{T}_x = T_k\} = \prod_{i=1}^{k-1} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) \mathbf{1}(B_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}),$$

for $k = 1$

$$\mathbf{1}\{\mathbb{T}_x \in (0, T_1)\} = \mathbf{1}(\{V_{t,x} \notin D \text{ for some } t \in (0, T_1)\})$$

and for $k \geq 2$

$$\mathbf{1}\{\mathbb{T}_x \in (T_{k-1}, T_k)\} = \prod_{i=1}^{k-1} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) \mathbf{1}(\{V_t^k \circ \theta_{T_{k-1}}(x) \notin D_{\delta_\varepsilon} \text{ for some } t \in (0, \tau_k)\}).$$

We choose $\kappa_\varepsilon := \lceil \frac{1}{h_\varepsilon} \rceil$. Hence

$$\begin{aligned}
& \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon \mathbb{T}_x} \mathbf{1}\{\mathbb{T}_x \in U\} \right] \\
& \leq \sum_{k=1}^{\kappa_\varepsilon-1} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon \mathbb{T}_x} (\mathbf{1}\{\mathbb{T}_x = T_k\} + \mathbf{1}\{\mathbb{T}_x \in (T_{k-1}, T_k)\}) \mathbf{1}(O_{\mathbb{T}_x, 0, x}(U)) \right] \\
& + \sum_{k=\kappa_\varepsilon}^{\infty} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon \mathbb{T}_x} \mathbf{1}\{\mathbb{T}_x \in (T_{k-1}, T_k)\} \right] \\
& =: S_1 + S_2 + S_3.
\end{aligned}$$

First we treat the easiest sum.

1) Estimate of S_3 : Due to $T_k = \tau_1 + \dots + \tau_k$ and the independence and stationarity of (τ_i) we obtain

$$S_3 \leq \sum_{k=\kappa_\varepsilon}^{\infty} \mathbb{E}[e^{-\theta Q h_\varepsilon T_1}]^k = \sum_{k=\kappa_\varepsilon}^{\infty} \frac{1}{(1 + \frac{\theta Q h_\varepsilon}{\beta_\varepsilon})^k} = \sum_{k=\kappa_\varepsilon}^{\infty} e^{k \ln(1 - \frac{\theta Q h_\varepsilon}{\beta_\varepsilon})}.$$

There is $\varepsilon_0 \in (0, 1)$ such that $\varepsilon \in (0, \varepsilon_0]$

$$S_3 \leq \sum_{k=\kappa_\varepsilon}^{\infty} e^{-k 2 \frac{\theta Q h_\varepsilon}{\beta_\varepsilon}} = \frac{e^{-\kappa_\varepsilon 2 \frac{\theta Q h_\varepsilon}{\beta_\varepsilon}}}{1 - e^{-2 \frac{\theta Q h_\varepsilon}{\beta_\varepsilon}}} \leq \frac{2e^{-\kappa_\varepsilon 2 \frac{\theta Q h_\varepsilon}{\beta_\varepsilon}}}{2 \frac{\theta Q h_\varepsilon}{\beta_\varepsilon}} \leq \frac{C}{3}.$$

2) Estimate of S_1 : We continue

$$\begin{aligned}
S_1 & \leq \sum_{k=1}^{\kappa_\varepsilon} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T_k} \mathbf{1}\{\mathbb{T}_x = T_k\} \mathbf{1}(O_{T_k, 0, x}(U)) \right] \\
& \leq \sum_{k=1}^{\kappa_\varepsilon} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[\prod_{i=1}^{k-1} e^{-\theta Q h_\varepsilon \tau_i} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) \mathbf{1}(B_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}) \mathbf{1}(O_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}(U)) \right].
\end{aligned}$$

Exploiting the same reasoning as in inequality (28) with the strong Markov property of X^ε for the jump times $(T_k)_{k \geq 1}$ instead of Markov property at deterministic times $k\mathcal{T}$, and the independence and stationarity of the increments we estimate the k -th summand of S_1 by

$$\begin{aligned}
& \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[\prod_{i=1}^{k-1} e^{-\theta Q h_\varepsilon \tau_i} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) e^{-\theta Q h_\varepsilon \tau_{k-1}} \mathbf{1}(B_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}) \mathbf{1}(O_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}(U)) \right] \\
& \leq \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[\left(e^{-\theta Q h_\varepsilon \tau_1} \mathbf{1}(A_{\tau_1, 0, x}) \mathbf{1}\{V_{T_1, x} \in D_{\delta_\varepsilon}\} + \mathbf{1}\{V_{T_1, x} \notin D_{\delta_\varepsilon}\} \right) \right. \\
& \quad \left. \mathbb{E} \left[\prod_{i=2}^{k-1} e^{-\theta Q h_\varepsilon \tau_i} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) e^{-\theta Q h_\varepsilon \tau_{k-1}} \mathbf{1}(B_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}) \mathbf{1}(O_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}(U)) \mid \mathcal{F}_{T_1} \right] \right] \\
& \leq \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_x) \right]
\end{aligned}$$

$$\begin{aligned}
& \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[\prod_{i=1}^{k-2} e^{-\theta Q h_\varepsilon \tau_i} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) e^{-\theta Q h_\varepsilon \tau_{k-2}} \mathbf{1}(B_{\tau_{k-1}, T_{k-2}, X_{T_{k-2}, x}}) \mathbf{1}(O_{\tau_{k-1}, T_{k-2}, X_{T_{k-2}, x}}(U)) \right] \\
& + \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(V_{T_1, x} \notin D_{\delta_\varepsilon})
\end{aligned} \tag{31}$$

where we use the abbreviation

$$\begin{aligned}
A_x &= A_{T_1, 0, x} \\
B_x &= B_{T_1, 0, x} \\
O_x^U &= O_{T_1, 0, x}(U).
\end{aligned}$$

The recursion from $k-1$ to 1 leads to

$$\begin{aligned}
& \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[\prod_{i=1}^{k-1} e^{-\theta Q h_\varepsilon \tau_i} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) e^{-\theta Q h_\varepsilon \tau_{k-1}} \mathbf{1}(B_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}) \mathbf{1}(O_{\tau_k, T_{k-1}, X_{T_{k-1}, x}}(U)) \right] \\
& \leq \left(\sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_x) \right] \right)^{k-1} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(B_x) \mathbf{1}(O_x^U) \right] \\
& \quad + \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(V_{T_1, x} \notin D_{\delta_\varepsilon}) \sum_{j=0}^{k-2} \left(\sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_x) \right] \right)^j.
\end{aligned} \tag{32}$$

In the same way we estimate the k -th summand of S_2 for $k \geq 1$.

$$\begin{aligned}
& \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[\prod_{i=1}^{k-1} e^{-\theta Q h_\varepsilon \tau_i} \mathbf{1}(A_{\tau_i, T_{i-1}, X_{T_{i-1}, x}}) e^{-\theta Q h_\varepsilon \tau_{k-1}} \mathbf{1}(\{V_t^k \circ \theta_{T_{k-1}}(x) \in D_{\delta_\varepsilon}^c \cap U \text{ for some } t \in (0, \tau_k)\}) \right] \\
& \leq \left(\sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_x) \right] \right)^{k-1} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(\{V_{t,x}^\varepsilon \in D \cap U \text{ for some } t \in (0, T_1)\}) \right] \\
& \quad + \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(V_{T_1, x} \notin D_{\delta_\varepsilon}) \sum_{j=0}^{k-2} \left(\sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_x) \right] \right)^j.
\end{aligned}$$

We show now that the first sum can be estimated by $1/(1+\theta)$, the Laplace transform of $\text{EXP}(1)$ evaluated at θ , plus a small error and that both additional sums tend to zero if ε does so.

Starting with the first factor of the main sum we obtain

$$\sup_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_y) \right] \leq \sup_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} (1 - \mathbf{1}\{V_{T,y} + G(V_{T,y}, \varepsilon W) \in D^c\}) \right].$$

Proposition 3.1 and the independence of W from T and V ensure the existence $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$\sup_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_y) \right] \leq \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - (1 - C) \int_{\mathcal{A}} \mathbb{P}(v + G(v, \varepsilon W) \in D^c) P(dv) \right).$$

Since by definition

$$\mathbb{P}(v + G(v, \varepsilon W) \in D^c) = \frac{1}{\beta_\varepsilon} \nu \left(\frac{1}{\varepsilon} E^{D^c}(v) \right),$$

\mathcal{A} is compact and the distance $d(\mathcal{A}, \partial D) > 0$, the regular variation of ν implies the existence of a constant $\varepsilon_0 \in (0, 1)$ such that

$$\sup_{v \in \mathcal{A}} \left| \frac{\mathbb{P}(v + G(v, \varepsilon W) \in D^c)}{\frac{h_\varepsilon}{\beta_\varepsilon} \mu(E^{D^c}(v))} - 1 \right| \leq C, \quad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

Hence there is $\varepsilon_0 \in (0, 1)$ such for $\varepsilon \in (0, \varepsilon_0]$

$$\begin{aligned} \sup_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_y) \right] &\leq \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - (1 - C)^2 \frac{h_\varepsilon}{\beta_\varepsilon} \int_{\mathcal{A}} \mu(E^{D^c}(u)) P(du) \right) \\ &= \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - (1 - C)^2 \frac{Q h_\varepsilon}{\beta_\varepsilon} \right). \end{aligned} \quad (33)$$

The second factor of the main sum can be treated analogously and we obtain for sufficiently small $\varepsilon_0 \in (0, 1)$ that

$$\sup_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(B_y) \mathbf{1}(O_y^U) \right] \leq (1 + C)^2 \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{Q(U \cap D^c) h_\varepsilon}{\beta_\varepsilon} \quad (34)$$

for $\varepsilon \in (0, \varepsilon_0]$, where

$$Q(D^c \cap U) = \int_{\mathcal{A}} \mu(E^{D^c \cap U}(u)) P(du).$$

For the remainder sum we exploit Corollary 3.2 which yields a constant $C' > 0$ and $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$\sup_{y \in D_{\delta_\varepsilon}} \mathbb{P}(\mathbb{T}_y \in (0, T)) \leq \frac{C' e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2}$$

and obtain

$$\begin{aligned} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(V_{T_1, x} \notin D_{\delta_\varepsilon}) \sum_{k=1}^{k_\varepsilon-1} \sum_{j=0}^{k-2} \left(\sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_x) \right] \right)^j \\ \leq \frac{C' e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2} \sum_{k=1}^{k_\varepsilon-1} \sum_{j=0}^{k-2} \left(\frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - (1 - C)^2 \frac{Q h_\varepsilon}{\beta_\varepsilon} \right) \right)^j. \end{aligned}$$

Let us call

$$q_\varepsilon = \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - (1 - C)^2 \frac{Q h_\varepsilon}{\beta_\varepsilon} \right).$$

Then

$$\begin{aligned} \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(V_{T_1, x} \notin D_{\delta_\varepsilon}) \sum_{k=1}^{k_\varepsilon-1} \sum_{j=0}^{k-2} \left(\sup_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_x) \right] \right)^j \\ =: \frac{C' e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2} \sum_{k=0}^{k_\varepsilon-2} \frac{1 - q_\varepsilon^{k-1}}{1 - q_\varepsilon} \end{aligned}$$

$$\leq \frac{C' e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2} \frac{\kappa_\varepsilon}{1 - q_\varepsilon} \leq \frac{C}{3}. \quad (35)$$

Eventually inequalities (33), (34) and (35) combined imply the existence of $\varepsilon_0 \in (0, 1)$ such that

$$S_1 \leq (1 + C)^2 \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{Q(D^c \cap U) h_\varepsilon}{\beta_\varepsilon} \sum_{k=1}^{\infty} \left(\frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - \frac{Q h_\varepsilon}{\beta_\varepsilon} (1 - C)^2 \right) \right)^{k-1} + \frac{C}{3}$$

for $\varepsilon \in (0, \varepsilon_0]$.

3) Estimate of S_2 : For $k = 1$ we exploit Corollary 3.5, which yields a constant $C' > 0$ and $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$\begin{aligned} \sup_{y \in D_{\delta_\varepsilon}} \mathbb{P}(\{\mathbb{T}_y \in (0, T)\} \cap O_{\mathbb{T}_y, 0, x}(U)) &\leq \sup_{x \in D_{\delta_\varepsilon}} \mathbb{P}(t \in [0, T] : V_{t, x} \notin D_{\delta_\varepsilon} \cap U) \\ &\leq \frac{C' e^{-\delta_\varepsilon^{-1}}}{(\beta_\varepsilon \delta_\varepsilon)^2} \\ &\leq ((1 + C)^3 - (1 + C)^2) \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{Q h_\varepsilon}{\beta_\varepsilon}. \end{aligned}$$

The last estimate follows by the algebraic choice of δ_ε . and eventually with the help of estimate (35) of the remainder sum

$$S_2 \leq ((1 + C)^3 - (1 + C)^2) \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{Q(D^c \cap U) h_\varepsilon}{\beta_\varepsilon} \sum_{k=1}^{\infty} \left(\frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - (1 - C)^2 \frac{Q h_\varepsilon}{\beta_\varepsilon} \right) \right)^{k-1} + \frac{C}{3}$$

Conclusion: We infer that there is a sufficiently small constant $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$\begin{aligned} \sup_{x \in D_{\delta_\varepsilon}} \mathbf{E} \left[e^{-\theta Q h_\varepsilon \mathbb{T}_x} \mathbf{1}\{X_{\mathbb{T}_y, y}^\varepsilon \in U\} \right] &\leq S_1 + S_2 + S_3 \\ &\leq (1 + C)^3 \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{Q(D^c \cap U) h_\varepsilon}{\beta_\varepsilon} \sum_{k=1}^{\infty} \left(\frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - \frac{Q h_\varepsilon}{\beta_\varepsilon} (1 - C)^3 \right) \right)^{k-1} \\ &= \frac{(1 + C)^3 \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{Q(D^c \cap U) h_\varepsilon}{\beta_\varepsilon}}{1 - \left(\frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \left(1 - \frac{Q h_\varepsilon}{\beta_\varepsilon} (1 - C)^3 \right) \right)} + C \\ &= \frac{(1 + C)^3}{\theta + (1 - C)^3} \frac{Q(D^c \cap U)}{Q} + C. \end{aligned}$$

By an appropriate renaming of the constant C we close the proof. ■

4.2 The lower bound

Proposition 4.2 *Let the assumptions of Proposition (4.1) be satisfied. Then for all $\theta > 0$, $U \in \mathfrak{B}(\mathbb{R}^d)$ satisfying (30) and $C \in (0, 1)$ there is $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$ the first exit time*

$\mathbb{T}_y = \mathbb{T}_y(\varepsilon)$ satisfies

$$\inf_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q(D^c) h_\varepsilon \mathbb{T}_y} \mathbf{1}\{X_{\mathbb{T}_y, y}^\varepsilon \in U\} \right] \geq \frac{Q(D^c \cap U)}{Q(D^c)} \frac{1-C}{1+\theta+C}.$$

Proof: We keep the notation introduced in the proof of Proposition 4.1. We define the following events for $y \in D_{\delta_\varepsilon}$ and $t, s \geq 0$ by

$$A_{t,s,y}^- = \{X_{r,\cdot} \circ \theta_s(y) \in D \text{ for all } r \in [0, t) \text{ and } X_{t,\cdot} \circ \theta_s(y) \in D_{\delta_\varepsilon}\}$$

and the abbreviation

$$A_y^- = A_{T_1, 0, y}^-.$$

The identical strong Markov property estimates from below (31) and (32) as in the proof of Proposition 4.1 only with inverted inequalities and neglecting all the nonnegative error terms yields

$$\begin{aligned} & \inf_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon \mathbb{T}_y} \mathbf{1}\{X_{\mathbb{T}_y, y}^\varepsilon \in U\} \right] \\ & \geq \sum_{k=1}^{\infty} \left(\inf_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_y^-) \right] \right)^{k-1} \inf_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(B_y) \mathbf{1}(D_y^U) \right]. \end{aligned}$$

Proposition 3.1 yields a constant $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$

$$\inf_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(A_y^-) \right] \geq \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} (1 - (1+C)^2 \frac{Q h_\varepsilon}{\beta_\varepsilon} (D^c))$$

and

$$\inf_{y \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon T} \mathbf{1}(B_y) \mathbf{1}(D_y^U) \right] \geq (1-C)^2 \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{h_\varepsilon}{\beta_\varepsilon} Q(D^c \cap U).$$

This eventually implies a constant $\varepsilon_0 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_0]$ follows

$$\begin{aligned} & \inf_{x \in D_{\delta_\varepsilon}} \mathbb{E} \left[e^{-\theta Q h_\varepsilon \mathbb{T}_x} \mathbf{1}\{X_{\mathbb{T}_x, x}^\varepsilon \in U\} \right] \\ & \geq (1-C)^2 \frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} \frac{h_\varepsilon}{\beta_\varepsilon} Q(U \cap D^c) \sum_{k=1}^{\infty} \left(\frac{\beta_\varepsilon}{\theta Q h_\varepsilon + \beta_\varepsilon} (1 - (1+C)^2 \frac{h_\varepsilon}{\beta_\varepsilon} Q) \right)^{k-1} \\ & = \frac{Q(U \cap D^c)}{Q(D^c)} \frac{(1-C)^2}{\theta + (1+C)^2}. \end{aligned}$$

An appropriate renaming of the constant C finishes the proof. ■

Proof: (Theorem 2.1) For $\rho \in (0, \frac{1}{2})$ we define $\rho^\varepsilon = \varepsilon^{-\rho}$ and verify condition (16) for $p = 4$ and (24) for the choice of ρ^ε and δ_ε .

$$\frac{\varepsilon \rho^\varepsilon}{\delta_\varepsilon^{(p+1)/2}} = \varepsilon^{1-\rho-5/2} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0+.$$

Since for small ε the intensity $\beta_\varepsilon \approx_\varepsilon \varepsilon^{\alpha\rho} \ell(\frac{1}{\varepsilon\rho}) \mu(\mathcal{B}_1^c(0))$ is asymptotically dominated by a polynomial order just as δ_ε , the reasoning is reduced to the fact that the exponential convergence of $e^{-\delta_\varepsilon}$ dominates $(\delta_\varepsilon \beta_\varepsilon)^{-2}$ in the limit as $\varepsilon \rightarrow 0+$. This implies relation (24). Therefore the upper bound by Proposition 4.1 and the lower bound by Proposition 4.2 are satisfied, which yields the desired result. ■

References

- [1] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, second edition, 2009.
- [2] N. Berglund and Gentz B. On the noise-induced passage through an unstable periodic orbit I: Two-level model. *Journal of Statistical Physics*, 114(5–6):1577–1618, 2004.
- [3] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its applications*. Cambridge University Press, 1987.
- [4] A. Bovier, M. Eckhoff, V. Gaynard, and M. Klein. Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. *Journal of the European Mathematical Society*, 6(4):399–424, 2004.
- [5] S. Brassesco. Some results on small random perturbations of an infinite dimensional dynamical system. *Stochastic Processes and their Applications*, 38:33–53, 1991.
- [6] S. Brassesco. Unpredictability of an exit time. *Stochastic Processes and their Applications*, 63:55–65, 1996.
- [7] M. V. Day. On the exponential exit law in the small parameter exit problem. *Stochastics*, 8:297–323, 1983.
- [8] M. V. Day. Exit cycling for the Van der Pol oscillator and quasipotential calculations. *Journal of Dynamics and Differential Equations*, 8(4):573–601, 1996.
- [9] A. Debussche, M. Högele, and P. Imkeller. *Metastability of reaction diffusion equations with small regularly varying noise*. Lecture Notes in Mathematics. Springer–Verlag, 2013. To appear.
- [10] L. N. Epele, H. Fanchiotti, A. Spina, and H. Vucetich. Noise-driven self-excited oscillators: Diffusion between limit cycles. *Physical Review A*, 31(4):2631–2638, 1985.
- [11] G. W. Faris and G. Jona-Lasinio. Large fluctuations for a nonlinear heat equation with noise. *Journal of Physics A: Mathematical and General*, 15(10):3025, 1982.
- [12] M. I. Freidlin. Random perturbations of reaction-diffusion equations: the quasideterministic approximation. *Transactions of the American Mathematical Society*, 305(2):665–697, 1988.
- [13] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften*. Springer, second edition, 1998.
- [14] V. V. Godovanchuk. Asymptotic probabilities of large deviations due to large jumps of a Markov process. *Theory of Probability and its Applications*, 26:314–327, 1982.
- [15] A. Goldbeter and F. Moran. Dynamics of a biochemical system with multiple oscillatory domains as a clue for multiple modes of neuronal oscillations. *European Biophysics Journal*, 15:277–287, 1988.
- [16] J. M. Hill, N. G. Lloyd, and J. M. Pearson. Limit cycles of a predator–prey model with intratrophic predation. *Journal of mathematical analysis and applications*, 349(2):544–555, 2009.

- [17] H. Hult and F. Lindskog. Regular variation for measures on metric spaces. *Publications de l'Institut Mathématique (Beograd). Nouvelle Série*, 80(94):121–140, 2006.
- [18] P. Imkeller and I. Pavlyukevich. First exit times of SDEs driven by stable Lévy processes. *Stochastic Processes and their Applications*, 116(4):611–642, 2006.
- [19] P. Imkeller, I. Pavlyukevich, and M. Stauch. First exit times of non-linear dynamical systems in \mathbb{R}^d perturbed by multifractal Lévy noise. *Journal of Statistical Physics*, 141(1):94–119, 2010.
- [20] O. Kallenberg. *Foundations of modern probability*. Probability and Its Applications. Springer, second edition, 2002.
- [21] H. A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, 7:284–304, 1940.
- [22] C. Kurrer and K. Schulten. Effect of noise and perturbations on limit cycle systems. *Physica D*, 50:311–320, 1991.
- [23] J. Liu and J. W. Crawford. Stability of an autocatalytic biochemical system in the presence of noise perturbations. *IMA Journal of Mathematics Applied in Medicine and Biology*, 15(4):339–350, 1998.
- [24] I. Pavlyukevich. First exit times of solutions of stochastic differential equations driven by multiplicative Lévy noise with heavy tails. *Stochastics and Dynamics*, 11(2&3), 2011.
- [25] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics*. Springer, second edition, 2004.
- [26] S. Resnick. On the foundations of multivariate heavy-tail analysis. *Journal of Applied Probability*, 41A:191–212, 2004.
- [27] Y. A. Saet and G. Viviani. The stochastic process of transitions between limit cycles for a special class of self-oscillators under random perturbations. *IEEE Transactions on Circuits and Systems*, CAS-34(6):691–695, 1987.
- [28] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.
- [29] R. Temam. *Infinite-dimensional dynamical systems in Mechanics and Physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, second edition, 1997.
- [30] A. D. Ventsel' and M. I. Freidlin. On small random perturbations of dynamical systems. *Russian Mathematical Surveys*, 25(1):1–55, 1970.
- [31] A. D. Wentzell. *Limit theorems on large deviations for Markov stochastic processes*, volume 38 of *Mathematics and Its Applications (Soviet Series)*. Kluwer Academic Publishers, 1990.